J.K. SHAH TEST SERIES

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SUGGESTED SOLUTION

SYJC - 2020

SUBJECT-MATHEMATICS AND STATISTICS

Test Code –SYJ 6045 A

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A.1.

(A) Let $y = xe^{-x} + 7$

Differentiating both sides with respect to *x*, we have

$$\frac{dy}{dx} = e^{-x} - xe^{-x}$$

By derivative of inverse function

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dy}{dx}\right)}, \text{ where } \frac{dy}{dx} \neq 0$$

$$= \frac{1}{e^{-x}(1-x)} (\text{whenever } x \neq 1)$$

$$= \frac{e^{x}}{1-x}$$

(B) Let $y = \sin^{-1}(\cos x)$

$$y = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right)$$

$$= \frac{\pi}{2} - x$$

Differentiating both sides with respect to *x*, we get

$$\frac{dy}{dx} = -1$$

(C) $x^3 + y^2 + xy = 7$

Differentiating both sides w.r.t. *x*, we get,

$$3x^2 + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$\therefore 3x^2 + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y \text{ x } 1 = 0$$

$$\therefore (2y + x)\frac{dy}{dx} = -3x^2 - y = -(y + 3x^2)$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y + 3x^2}{2y + x}\right).$$

(D) $y = 2^{3x} \sin^5 x \cdot \log x$

Taking logarithm of both the sides, we get

$\log y = \log [2^{3x} \sin^5 x \cdot \log x]$

$= 3x \log 2 + 5 \log \sin x + \log \log x$

Differentiating both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = 3 \cdot \log 2 + \frac{5\cos x}{\sin x} + \frac{\frac{1}{x}}{\log x}$$

$$\frac{dy}{dx} = 2^{3x} \sin^5 x \cdot \log x \times \left[3\log 2 + 5\cot x + \frac{1}{x\log x}\right]$$

(E) $e^x + e^y = e^{x-y}$

Differentiating both sides w.r.t. x, we get,

$$e^x + e^y \frac{dy}{dx} = e^{x-y} \cdot \frac{d}{dx}(x - y)$$

$$\therefore \quad e^x + e^y \frac{dy}{dx} = e^{x-y}\left(1 - \frac{dy}{dx}\right)$$

$$\therefore \quad e^x + e^y \frac{dy}{dx} = e^{x-y} - e^{x-y}\frac{dy}{dx}$$

$$\therefore \quad (e^y + e^{x-y})\frac{dy}{dx} = e^{x-y} - e^x$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{x-y} - e^x}{e^y + e^{x-y}}.$$

(F) $x^3y^3 = x^2 - y^2$

Differentiating both sides w.r.t. x, we get,

$$x^3 \frac{d}{dx}(y^3) + y^3 \frac{d}{dx}(x^3) = 2x - 2y\frac{dy}{dx}$$

$$\therefore \quad x^3 \times 3y^2 \frac{dy}{dx} + y^3 \times 3x^2 = 2x - 2y\frac{dy}{dx}$$

$$\therefore \quad 3x^3y^2 \frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 3x^2y^3$$

$$\therefore \quad (3x^3y^2 + 2y)\frac{dy}{dx} = 2x - 3x^2y^3$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x - 3x^2y^3}{3x^3y^2 + 2y} = \frac{x(2 - 3xy^3)}{y(3x^3y + 2)}$$

$$= \frac{x}{y}\left(\frac{2 - 3xy^3}{2 + 3x^3y}\right).$$

A.2.

(A) $x = \left(u + \frac{1}{u}\right)^2, \; y = 2^{\left(u + \frac{1}{u}\right)}$

Differentiating x and y w.r.t. u, we get,

$$\frac{dx}{du} = \frac{d}{du}\left(u + \frac{1}{u}\right)^2 = 2\left(u + \frac{1}{u}\right) . \frac{d}{du}\left(u + \frac{1}{u}\right)$$

$$= 2\left(u+\frac{1}{u}\right)\left(1-\frac{1}{u^2}\right)$$

And $\frac{dy}{du} = \frac{d}{du}\left[2^{\left(u+\frac{1}{u}\right)}\right]$

$$= 2^{\left(u+\frac{1}{u}\right)}.\log 2.\frac{d}{du}\left(u+\frac{1}{u}\right)$$

$$= 2^{\left(u+\frac{1}{u}\right)}.\log 2.\left(1-\frac{1}{u^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{(dy/du)}{(dx/du)} = \frac{2^{\left(u+\frac{1}{u}\right)}.\log 2.\left(1-\frac{1}{u^2}\right)}{2\left(u+\frac{1}{u}\right)\left(1-\frac{1}{u^2}\right)}$$

$$= \frac{2^{\left(u+\frac{1}{u}\right)}.\log 2}{2\left(u+\frac{1}{u}\right)}$$

$$= \frac{y\log 2}{2\sqrt{x}}$$

(B) $y = \tan^{-1}\left(\frac{x+\sqrt{x}}{1-\sqrt{x^3}}\right)$

$$= \tan^{-1}\left(\frac{x+\sqrt{x}}{1-x\sqrt{x}}\right)$$

$$= \tan^{-1}x + \tan^{-1}\sqrt{x}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}x\right) + \frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)$$

$$= \frac{1}{1+x^2} + \frac{1}{1+\left(\sqrt{x}\right)^2}.\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x} \text{ x } \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{1+x^2} + \frac{1}{2\sqrt{x}\,(1+x)}$$

(C) g = t^2 + 2t + 1, h = tan $\left(\frac{\pi t}{4}\right)$ $-$cos $\left(\frac{\pi t}{2}\right)$

Differentiating g and h w.r.t. *t*, we get,

$\frac{dg}{dt} = \frac{d}{dt}(t^2+2t+1)$ = 2t + 2 × 1 + 0 = 2t + 2

and $\frac{dh}{dt} = \frac{d}{dt}\left[tan\left(\frac{\pi t}{4}\right) - cos\left(\frac{\pi t}{2}\right)\right]$

$= \sec^2\left(\frac{\pi t}{4}\right) \cdot \frac{d}{dt}\left(\frac{\pi t}{4}\right) - \left[-sin\left(\frac{\pi t}{2}\right)\right] \cdot \frac{d}{dt}\left(\frac{\pi t}{2}\right)$

$= \sec^2\left(\frac{\pi t}{4}\right) \times \frac{\pi}{4} + sin\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2}$

$= \frac{\pi}{4} \sec^2\left(\frac{\pi t}{4}\right) + \frac{\pi}{2} \sin\left(\frac{\pi t}{2}\right)$

$\therefore \frac{dg}{dh} = \frac{(dg/dt)}{(dh/dt)} = \frac{2t+2}{\frac{\pi}{4}sec^2\left(\frac{\pi t}{4}\right)+\frac{\pi}{2}sin\left(\frac{\pi t}{2}\right)}$

$\therefore \left(\frac{dg}{dh}\right)_{at\ t=1} = \frac{2(1)+2}{\frac{\pi}{4}sec^2\frac{\pi}{4}+\frac{\pi}{2}\sin\frac{\pi}{2}}$

$= \frac{4}{\frac{\pi}{4}(\sqrt{2})^2+\frac{\pi}{2}\times 1} = \frac{4}{\frac{\pi}{2}+\frac{\pi}{2}}$

$= \frac{4}{\pi}.$

(D) x = $\sec^{-1}\sqrt{1+t^2}$, y = $\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

Put t = tan θ. Then θ = $\tan^{-1} t$

$\therefore$ x = $\sec^{-1}\sqrt{1+tan^2\theta}$, y = $\cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$

$\therefore$ x = $\sec^{-1}\sqrt{sec^2\theta}$, y = $\cos^{-1}$ (cos 2θ)

$\therefore$ x = $\sec^{-1}$ (sec θ) = θ, y = 2θ

$\therefore$ x = $\tan^{-1} t$, y = $2\tan^{-1} t$

Differentiating x and y w.r.t. *t*, we get,

$\frac{dx}{dt} = \frac{d}{dt}(tan^{-1} t) = \frac{1}{1+t^2}$

and $\frac{dy}{dx} = 2\frac{d}{dt}(\tan^{-1} t) = 2 \times \frac{1}{1+t^2} = \frac{2}{1+t^2}$

$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{2}{1+t^2}\right)}{\left(\frac{1}{1+t^2}\right)} = 2.$

A.3

(A) $y = x^x + (7x-1)^x$

$\therefore\ y = u + v$

Differentiating both sides with respect to *x*,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots \qquad (1)$$

$$u = x^x$$

Taking logarithm of both the sides,

Log $u = x \log x$

Differentiating both sides with respect to x,

$$\frac{1}{u} \quad \frac{du}{dx}=1.\log x + x.\frac{1}{x}$$

$$\frac{du}{dx}= x^x(\log x+1) \qquad \ldots \qquad (2)$$

As $v=(7x-1)^x$,

Taking logarithm of both the sides,

$$\log v = x\log(7x-1)$$

Differentiating both sides with respect to x,

$$\frac{1}{v}\frac{dv}{dx}=1.\log(7x-1)+x.\frac{7}{(7x-1)}$$

$$\frac{dv}{dx}=(7x-1)^x \text{ x } \left(\log(7x-1)+\frac{7x}{7x-1}\right) \qquad \ldots \qquad (3)$$

Substituting (2) and (3) in (1) we get

$$\frac{dy}{dx} = x^x(\log \ x+1)+$$

$$(7x-1)^x \text{ x } \left(\log(7x-1)+\frac{7x}{7x-1}\right)$$

(B) $y=5^x.e^{\cos x}.\tan^{-1}x$

$$\therefore \log y = \log\left(5^x.e^{\cos x}.\tan^{-1}x\right)$$

$$= \log 5^x+\log e^{\cos x}+\log\left(\tan^{-1}x\right)$$

$$= x\log 5+\cos x.\log e+\log\left(\tan^{-1}x\right)$$

$$= x\log 5+\cos x+\log\left(\tan^{-1}x\right) \ \ldots \left[\because \log e=1\right]$$

Differentiating both sides w.r.t. x, we get,

$$\frac{1}{y}.\frac{dy}{dx} = (\log 5).\frac{d}{dx}(x)+\frac{d}{dx}(\cos x)+\frac{d}{dx}\left[\log\left(\tan^{-1}x\right)\right]$$

$$= (\log 5) \text{ x } 1+(-\sin x)+\frac{1}{\tan^{-1}x}.\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$= \log 5-\sin x+\frac{1}{\tan^{-1}x} \text{ x } \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = y\left[\log 5-\sin x+\frac{1}{\left(1+x^2\right)\tan^{-1}x}\right]$$

$$= 5^x . e^{\cos x} . \tan^{-1} x \left[\log 5 - \sin x + \frac{1}{\left(1+x^2\right)\tan^{-1} x} \right]$$

(C) $y = x^x + (\tan x)^x$

Put $u = x^x$ and $v = (\tan x)^x$.

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{.....(1)}$$

Take $u = x^x$

$\therefore \log u = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get,

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{x} + (\log x)(1) = 1 + \log x$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x [1 + \log x] \qquad \text{.....(2)}$$

Also, $v = (\tan x)^x$

$\therefore \log v = \log (\tan x)^x = x \cdot \log \tan x$

Differentiating both sides w.r.t. x, we get,

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}(x \cdot \log \tan x)$$

$$= x \cdot \frac{d}{dx}(\log \tan x) + (\log \tan x) \cdot \frac{d}{dx}(x).$$

$$= x \times \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x) + (\log \tan x) \times 1$$

$$= x \times \frac{1}{\tan x} \times sec^2 x + \log \tan x$$

$$\therefore \frac{dv}{dx} = v\left[x \times \frac{\cos x}{\sin x} \times \frac{1}{cos^2 x} + \log \tan x\right]$$

$$= (\tan x)^x [2x \operatorname{cosec} 2x + \log \tan x] \qquad \text{.......(3)}$$

From (1), (2) and (3), we get,

$\frac{dy}{dx} = x^x$ (1 + log x) + (tan x)^x (2xcosec 2x+ log tan x).

(D) tan $\left(\frac{x+y}{x-y}\right) = a$

$\therefore \frac{x+y}{x-y} = \tan^{-1} a = p$ ….(Say)

$\therefore$ x + y = $px - py$

$\therefore$ y + $py = px - x$

$\therefore$ (1 + p)y = (P – 1) x

$\therefore \frac{y}{x} = \frac{p-1}{1+p}$ ….(A constant)

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$

$\therefore \frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2} = 0$

$\therefore x\frac{dy}{dx} - y \times 1 = 0$

$\therefore x\frac{dy}{dx} = y$

$\therefore \frac{dy}{dx} = \frac{y}{x}.$

Alternative Method :

tan $\left(\frac{x+y}{x-y}\right) = a$

$\therefore \frac{x+y}{x-y} = \tan^{-1}a$

Differentiating both sides w.r.t. x, we get

$\frac{d}{dx}\left(\frac{x+y}{x-y}\right) = \frac{d}{dx}(tan^{-1}a)$

$\therefore \frac{(x-y)\cdot\frac{d}{dx}(x+y) - (x+y)\cdot\frac{d}{dx}(x-y)}{(x-y)^2} = 0$

$\therefore (x-y)\left(1 + \frac{dy}{dx}\right) - (x+y)\left(1 - \frac{dy}{dx}\right) = 0$

$\therefore$ (x – y) + (x – y) $\frac{dy}{dx}$ – (x + y) + (x + y) $\frac{dy}{dx}$ = 0

$\therefore$ (x – y + x + y) $\frac{dy}{dx}$ = – x + y + x + y

$\therefore$ 2x $\frac{dy}{dx} = 2y$

$\therefore$ $\frac{dy}{dx} = \frac{y}{x}$.